DAY TWENTY SIX

The Circle

Learning & Revision for the Day

- Concept of Circle
- Line and Circle
- Equation of Tangents
- Equation of Normal
- Pair of Tangents
- Common Tangents of Two Circles
- Director Circle
- Chord of Contact
- Pole and Polar
- Angle of Intersection of Two Circles
- Family of Circles
- Radical Axis
- Coaxial System of Circles

Concept of Circle

Circle is the locus of a point which moves in a plane, such that its distance from a fixed point in the plane is a constant. The fixed point is the **centre** and the constant distance is the **radius**.

Standard Form of Equation of a Circle

The equation of a circle whose centre is at (h, k) and radius r is given, is $(x-h)^2+(y-k)^2=r^2$. It is also known as the central form of the equation of a circle.

The equation $x^2+y^2+2gx+2fy+c=0$ always represents a circle, whose centre is $(-g,-f)$ and radius is $\sqrt{g^2+f^2-c}$. This is known as the general equation of a circle.

Equation of Circle when the End Points of a Diameter are Given

If A and B are end points of a diameter of a circle whose coordinates are (x_1, y_1) and (x_2, y_2), respectively.
Then, the equation of circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

NOTE Parametric equations of a circle is $(x-h)^2+(y-k)^2=r^2$, where $x=h+r\cos\theta$, $y=k+r\sin\theta$, $0\le\theta\le 2\pi$.

Intercept on Axes

The length of intercepts made by the circle $x^2+y^2+2gx+2fy+c=0$ with X and Y-axes are $2\sqrt{g^2-c}$ and $2\sqrt{f^2-c}$, respectively.

Position of a Point w.r.t. a Circle

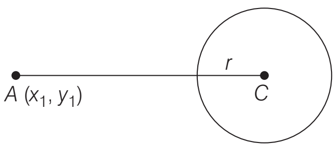
A point (x_1, y_1) lies outside, on or inside a circle

$$S = x^2 + y^2 + 2gx + 2fy + c = 0$$

according as $S_1 >, =,$ or < 0,

where, $\quad S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Greatest and **least distance** of a point $A(x_1, y_1)$ from a circle with centre C and radius r as shown in the figure below, is $|AC + r|$ and $|AC - r|$.



Line and Circle

Let $y = mx + c$ be a line and $x^2 + y^2 = r^2$ be a circle. If r is the radius of circle and $p = \left|\dfrac{c}{\sqrt{1+m^2}}\right|$ is the length of the perpendicular from the centre on the line, then

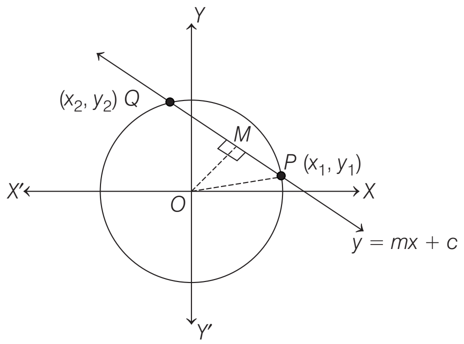
(i) $p > r \Leftrightarrow$ the line passes outside the circle.

(ii) $p = r \Leftrightarrow$ the line touches the circle or the line is a tangent to the circle.

(iii) $p < r \Leftrightarrow$ the line intersect the circle at two points or the line is secant of the circle.

(iv) $p = 0 \Leftrightarrow$ the line is a diameter of the circle.

- The length of the intercept cut-off from the line $y = mx + c$ by the circle $x^2 + y^2 = r^2$ is



$$PQ = 2\sqrt{\frac{r^2(1+m^2) - c^2}{1+m^2}}$$

Equation of Tangents

A line which touch only one point of a circle is called its tangent as shown in the following figure. This tangent may be in slope or point form as given below.

1. Slope Form

(i) The equation of tangents of slope m to the circle $(x-a)^2 + (y-b)^2 = r^2$ are given by

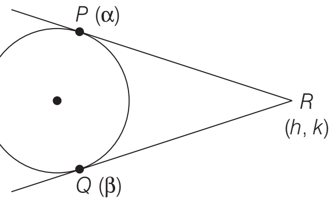
$$y - b = m(x - a) \pm r\sqrt{1+m^2}$$

and the coordinates of the points of contact are

$$\left(a \pm \frac{mr}{\sqrt{1+m^2}}, b \mp \frac{r}{\sqrt{1+m^2}}\right).$$

(ii) Point of intersection of the tangent drawn to the circle $x^2 + y^2 = r^2$ at the point $P(\alpha)$ and $Q(\beta)$ is

$$h = \frac{r\cos\dfrac{\alpha+\beta}{2}}{\cos\dfrac{\alpha-\beta}{2}} \text{ and } k = \frac{r\sin\dfrac{\alpha+\beta}{2}}{\cos\dfrac{\alpha-\beta}{2}}.$$



2. Point Form

(i) Equation of tangent for $x^2 + y^2 = r^2$ at (x_1, y_1) is $xx_1 + yy_1 = r^2$.

(ii) The equation of the tangent at the point $P(x_1, y_1)$ to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

3. Parametric form

(i) Parametric coordinates of circle $x^2 + y^2 = r^2$ is $(r\cos\theta, r\sin\theta)$, then equation of tangent at $(r\cos\theta, r\sin\theta)$ is $x\cos\theta + y\sin\theta = r$.

(ii) Equation of the tangent to the circle $(x-h)^2 + (y-k)^2 = r^2$ at $(h + r\cos\theta, k + r\sin\theta) = r^2$ is $(x-h)\cos\theta + (y-k)\sin\theta = r$.

Length of the Tangents

The length of the tangent from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$$

Equation of Normal

The normal at any point on a curve is a straight line which is perpendicular to the tangent to the curve at that point.

1. Point Form

The equation of normal to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ or } x^2 + y^2 = a^2$$

at any point (x_1, y_1) is $\dfrac{x - x_1}{x_1 + g} = \dfrac{y - y_1}{y_1 + f}$

or $\qquad \dfrac{x}{x_1} = \dfrac{y}{y_1}$

2. Parametric Form

The equation of normal to the circle $x^2 + y^2 = a^2$ at point $(a\cos\theta, a\sin\theta)$ is $\dfrac{x}{\cos\theta} = \dfrac{y}{\sin\theta}$ or $y = x\tan\theta$.

Pair of Tangents

From a given point, two tangents can be drawn to a circle which are real and distinct, coincident or imaginary according as the given point lies outside, on or inside the circle.

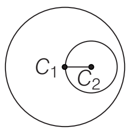
The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is $SS_1 = T^2$.

where, $S = x^2 + y^2 - a^2, S_1 = x_1^2 + y_1^2 - a^2$ and $T = xx_1 + yy_1 - a^2$

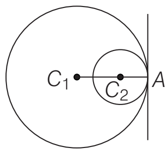
Common Tangents of Two Circles

Let the centres and radii of two circles be C_1, C_2 and r_1, r_2, respectively.

(i) When one circle contains other as shown in the figure below, no common tangent is possible. Condition $C_1C_2 < r_1 - r_2$.

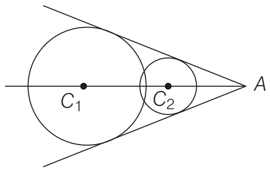


(ii) When two circles touch internally as shown in the figure, one common tangent is possible.



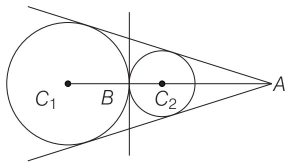
Condition, $C_1C_2 = r_1 - r_2$

(iii) When two circles intersect as shown in the figure below, two common tangents are possible.



Condition, $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

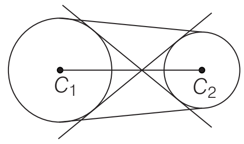
(iv) When two circles touch externally as shown in the figure below, three common tangents are possible.



Condition, $C_1C_2 = r_1 + r_2$

A divides C_1C_2 externally in the ratio $r_1 : r_2$. **B** divides C_1C_2 internally in the ratio $r_1 : r_2$.

(v) When two circles are separately as shown in the figure below, four common tangents are possible.



Condition, $C_1C_2 > r_1 + r_2$

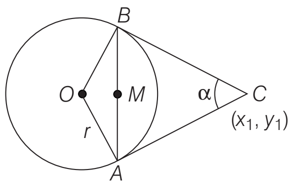
Director Circle

The locus of the point of intersection of two perpendicular tangents to a given circle is known as its director circle. The equation of the director circle of the circle

$$x^2 + y^2 = a^2 \text{ is } x^2 + y^2 = 2a^2.$$

Chord of Contact

- The chord joining the points of contact of the two tangents from a point, which is outside is called the chord of contact of tangents.
- The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$ or $T = 0$.
- If AB is a chord of contact of tangents from C to the circle $x^2 + y^2 = r^2$ and M is the mid-point of AB as shown in figure. Then,



Angle between two tangents $\angle ACB$ is $2\tan^{-1}\dfrac{r}{\sqrt{S_1}}$.

Chord Bisected at a Given Point

The equation of the chord of the circle

$$x^2 + y^2 = a^2$$

bisected at the point (x_1, y_1) is given by

$$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$$

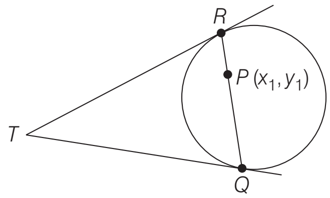
or $$T = S_1$$

Of all the chords which passes through a given point $M(a,b)$ inside the circle the shortest chord is one whose middle point is (a,b).

Pole and Polar

If through a point $P(x_1, y_1)$ (inside or outside a circle) there be drawn any straight line to meet the given circle at Q and R as shown in the following figure, the locus of the point of intersection of the tangents at Q and R is called the **polar** of P and P is called the **pole** of the polar.

The polar of a point $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 = a^2$ as shown in the below figure is $xx_1 + yy_1 = a^2$ or $T = 0$.



1. Conjugate Points

Two points A and B are conjugate points with respect to a given circle, if each lies on the polar of the other with respect to the circle.

2. Conjugate Lines

If two lines be such that the pole of one line lies on the other, then they are called conjugate lines with respect to the given circle.

Angle of Intersection of Two Circles

The angle of intersection of two circles is defined as the angle between the tangents to the two circles at their point of intersection is given by $\cos\theta = \dfrac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}$

where, d is distance between centres of the circles.

Orthogonal Circles

Two circles are said to be intersect orthogonally, if their angle of intersection is a right angle.

(Radius of Ist circle)^2 + (Radius of IInd circle)^2

= (Distance between centres)^2

$\Rightarrow \quad 2(g_1g_2 + f_1f_2) = c_1 + c_2$

which is the condition of orthogonality of two circles. The circles having radii r_1 and r_2 intersect orthogonally. Then, length of their common chord is $\dfrac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$.

Family of Circles

1. The equation of a family of circles passing through the intersection of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and line $L \equiv lx + my + n = 0$ is
   $x^2 + y^2 + 2gx + 2fy + c + \lambda\,(lx + my + n) = 0$ or $S + \lambda L = 0$
   where, λ is any real number.
2. The equation of the family of circles passing through the point $A(x_1, y_1)$ and $B(x_2, y_2)$ is
   $$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$
   $$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda L = 0$$
   where, $L = 0$ represents the line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$ and $\lambda \in R$.
3. The equation of the family of circles touching the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ at point $P(x_1, y_1)$ is
   $$x^2 + y^2 + 2gx + 2fy + c + \lambda\{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\} = 0$$
   or $\quad S + \lambda L = 0$
   where, $L = 0$ is the equation of the tangent to $S = 0$ at (x_1, y_1) and $\lambda \in R$.
4. The equation of a family of circles passing through the intersection of the circles
   $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$
   and $\quad S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is
   $S_1 + \lambda S_2 = 0$, where $(\lambda \neq -1)$ is an arbitrary real number.

Radical Axis

The radical axis of two circles is the locus of a point which moves in such a way that the lengths of the tangents drawn from it to the two circles are equal.

The radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is given by, $S_1 - S_2 = 0$.

(i) The equations of radical axis and the common chord of two circles are identical.

(ii) The radical axis of two circles is always perpendicular to the line joining the centres of the circles.

(iii) **Radical centre** The point of intersection of radical axis of three circles whose centres are non-collinear, taken in pairs is called their radical centre.

Coaxial System of Circles

A system of circles is said to be coaxial system of circles, if every pair of the circles in the system has the same radical axis.

1. If the equation of a member of a system of coaxial circles is $S = 0$ and the equation of the common radical axis is $L = 0$, then the equation representing the coaxial system of circle is $S + \lambda L = 0$, where $\lambda \in R$.
2. If $S_1 = 0$ and $S_2 = 0$ are two circles, then $S_1 + \lambda S_2 = 0$
   $S_1 + \lambda(S_1 - S_2) = 0$ or $S_2 + \lambda(S_1 - S_2) = 0, \lambda \in R$
   represent a family of coaxial circles having $S_1 - S_2 = 0$ as the common radical axis.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 Equation of a circle whose two diameters are along the lines $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ and passes through the origin is

(a) $x^2 + y^2 + 2x - 44y = 0$
(b) $17x^2 + 17y^2 - 2x + 44y = 0$
(c) $17x^2 + 17y^2 + 2x - 44y = 0$
(d) None of the above

2 Points (2, 0), (0, 1), (4, 5), and (0, a) are concyclic. Then a is equal to

(a) $\frac{14}{3}$ or 1 (b) 14 or $\frac{1}{3}$
(c) $-\frac{14}{3}$ or -1 (d) None of these

3 The abscissae of two points A and B are the roots of the equation $x^2 + 2ax - 4 = 0$ and their ordinates are the roots of the equation $x^2 + 2bx - 9 = 0$. Then equation of the circle with AB as diameter is

(a) $x^2 + y^2 - ax - bx + 13 = 0$
(b) $x^2 + y^2 + ax + by - 13 = 0$
(c) $x^2 + y^2 + 2ax + 2by - 13 = 0$
(d) None of the above

4 If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points, then

(a) $a_1a_2 = b_1b_2$ (b) $a_1b_1 = a_2b_2$
(c) $a_1b_1 = a_2b_1$ (d) None of these

5 The equation of circle which passes through the points (2, 0) and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$ as $c \to 1$ is

(a) $25(x^2 + y^2) - 20x + 2y + 60 = 0$
(b) $25(x^2 + y^2) - 20x + 2y - 60 = 0$
(c) $25(x^2 - y^2) - 20x - 2y - 60 = 0$
(d) None of the above

6 The lines $3x - y + 3 = 0$ and $x - 3y - 6 = 0$ cut the coordinate axes at concyclic points. The equation of the circle through these points is

(a) $x^2 + y^2 - 5x - y - 6 = 0$ (b) $x^2 + y^2 + 5x + y + 6 = 0$
(c) $x^2 + y^2 + 2xy = 0$ (d) None of these

7 The circle passing through $(1, -2)$ and touching the X-axis to at $(3, 0)$ also passes through the point

→ **JEE Mains 2013**

(a) $(-5, 2)$ (b) $(2, -5)$
(c) $(5, -2)$ (d) $(-2, 5)$

8 AB is chord of the circle $x^2 + y^2 = 25$. The tangents of A and B intersect at C. If $(2, 3)$ is the mid-point of AB, then area of the quadrilateral $OACB$ is

(a) $50\sqrt{\frac{13}{3}}$ (b) $50\sqrt{\frac{3}{13}}$ (c) $50\sqrt{3}$ (d) $\frac{50}{\sqrt{3}}$

9 Two vertices of an equilateral triangle are $(-1, 0)$ and $(1, 0)$ and the third vertex lies above the X-axis. Find the equation of its circumcircle.

(a) $x^2 - y^2 + \frac{2y}{\sqrt{3}} + 1 = 0$ (b) $x^2 + y^2 - \frac{2y}{\sqrt{3}} - 1 = 0$
(c) $x^2 - y^2 - \frac{y}{\sqrt{3}} = 0$ (d) None of these

10 Circles are drawn through the point (2, 0) to cut intercept of length 5 units on the X-axis. If their centres lie in the first quadrant, then their equation for $k > 0$ is

(a) $x^2 + y^2 - 9x + 2ky + 14 = 0$
(b) $3x^2 + 3y^2 + 27x - 2ky + 42 = 0$
(c) $x^2 + y^2 - 9x - 2ky + 14 = 0$
(d) $x^2 + y^2 - 2kx - 9y + 14 = 0$

11 Circles are drawn through the points (a, b) and $(b, -a)$ such that the chord joining the two points subtends an angle of 45° at any point of the circumference. Then, the distance between the centres is

(a) $\sqrt{3}$ times the radius of either circle
(b) 2 times the radius of either circle
(c) $\frac{1}{\sqrt{2}}$ times the radius of either circle
(d) $\sqrt{2}$ times the radius of either circle

12 Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. If the tangents at the points $B(1, 7)$ and $D(4, -2)$ on the circle meet at C, then find the area of the quadrilateral $ABCD$.

(a) 78 (b) 75 (c) 79 (d) 85

13 Find the equation of a circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has double of its area.

(a) $x^2 + y^2 - 6x + 12y - 15 = 0$
(b) $x^2 + y^2 - 6x - 12y + 15 = 0$
(c) $x^2 + y^2 - 6x + 12y + 15 = 0$
(d) None of the above

14 The equation of the locus of the mid-points of the chords of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtend an angle of $2\pi/3$ at its centre is

(a) $x^2 + y^2 + 3x - y + 31/16 = 0$
(b) $x^2 + y^2 - 3x + y + 31/16 = 0$
(c) $x^2 + y^2 - 3x + y - 31/16 = 0$
(d) None of the above

15 Let PQ and RS be tangents at the extremities of the diameter PR of the circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then $2r$ equals

(a) $\sqrt{(PQ \cdot RS)}$ (b) $(PQ + RS)/2$
(c) $2\,PQ \cdot RS/(PQ + RS)$ (d) $\sqrt{\left[\frac{(PQ^2 + RS^2)}{2}\right]}$

16 If the line $ax + by = 0$ touches the circle $x^2 + y^2 + 2x + 4y = 0$ and is normal to the circle $x^2 + y^2 - 4x + 2y - 3 = 0$, (a, b) is given by

(a) (2, 1) (b) (1,–2) (c) (1, 2) (d) (–1, 2)

17 A circle touches the hypotenuse of a right angle triangle at its middle point and passes through the mid-point of the shorter side. If a and b $(a < b)$ be the length of the sides, then the radius is

(a) $\frac{b}{a}\sqrt{a^2 + b^2}$ (b) $\frac{b}{2a}\sqrt{a^2 - b^2}$

(c) $\frac{b}{4a}\sqrt{a^2 + b^2}$ (d) None of these

18 The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ is **→ JEE Mains 2015**

(a) 1 (b) 2 (c) 3 (d) 4

19 If the two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then

(a) $2 < r < 8$ (b) $r < 2$
(c) $r = 2$ (d) $r > 2$

20 Let C be the circle with centre at (1, 1) and radius 1. If T is the circle centred at (0, k) passing through origin and touching the circle C externally, then the radius of T is equal to **→ JEE Mains 2014**

(a) $\frac{\sqrt{3}}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{1}{2}$ (d) $\frac{1}{4}$

21 The tangent to the circle $x^2 + y^2 = 5$ at the point (1, – 2), also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$, find its point of contact.

(a) $x = 2, y = 1$ (b) $x = 3, y = -1$
(c) $x = 5, y = 7$ (d) None of these

22 If the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$, then the value of c is **→ JEE Mains 2018**

(a) 195 (b) 185 (c) 85 (d) 95

23 For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents from the point $P(6, 8)$ to the circle and the chord of contact is maximum.

(a) $r = 4$ (b) $r = 5$ (c) $r = 3$ (d) $r = 1$

24 From any point on the circle $x^2 + y^2 = a^2$ tangents are drawn to the circle $x^2 + y^2 = a^2 \sin^2 \alpha$. The angle between them is

(a) $\alpha / 2$ (b) α (c) 2α (d) None of these

25 If one of the diameters of the circle, given by the equation $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at (–3, 2), then the radius of S is **→ JEE Mains 2016**

(a) $5\sqrt{2}$ (b) $5\sqrt{3}$ (c) 5 (d) 10

26 The length of the common chord of two circles $(x - a)^2 + (y - b)^2 = c^2$ and $(x - b)^2 + (y - a)^2 = c^2$ is

(a) $\sqrt{4c^2 + 2(a - b)^2}$ (b) $\sqrt{4c^2 - (a - b)^2}$
(c) $\sqrt{4c^2 - 2(a - b)^2}$ (d) $\sqrt{2c^2 - 2(a - b)^2}$

27 If P, Q and R are the centres and r_1, r_2 and r_3 are the corresponding radii of the three circles form a system of coaxial circle, then $r_1^2 \cdot QR + r_2^2 \cdot RP + r_3^2 \cdot PQ$ is equal to

(a) $PQ \cdot QR \cdot RP$ (b) $-PQ \cdot QR \cdot RP$
(c) $PQ + QR + RP$ (d) $\frac{PQ}{QR} \times RP$

28 The condition that the chord $x \cos \alpha + y \sin \alpha - p = 0$ of $x^2 + y^2 - a^2 = 0$ may subtend a right angle at the centre of circle, is

(a) $a^2 = 2p^2$ (b) $p^2 = 2a^2$ (c) $a = 2p$ (d) $p = 2a$

29 The equation of the circle of minimum radius which contains the three circles $x^2 + y^2 - 4y - 5 = 0$,
$x^2 + y^2 + 12x + 4y + 31 = 0$
and $x^2 + y^2 + 6x + 12y + 36 = 0$ is

(a) $\left(x - \frac{31}{18}\right)^2 + \left(y - \frac{23}{12}\right)^2 = \left(3 - \frac{5}{36}\sqrt{949}\right)^2$

(b) $\left(x + \frac{23}{12}\right)^2 + \left(y - \frac{31}{18}\right)^2 = \left(3 + \frac{5}{36}\sqrt{949}\right)^2$

(c) $\left(x + \frac{31}{18}\right)^2 + \left(y + \frac{23}{12}\right)^2 = \left(3 + \frac{5}{36}\sqrt{949}\right)^2$

(d) None of the above

30 The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid-point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is

(a) 16 (b) 4 (c) 8 (d) 2

Direction (Q. Nos. 31-35) *Each of these questions contains two statements : Statement* I *(Assertion) and Statement* II *(Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (d) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

31 Tangents are drawn from the point (17, 7) to the circle $x^2 + y^2 = 169$.

Statement I The tangents are mutually perpendicular.

Statement II The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$.

32 Consider the radius should be zero in limiting points.

Statement I Equation of a circle through the origin and belonging to the coaxial system, of which the limiting points are $(1, 1)$ and $(3, 3)$ is

$2x^2 + 2y^2 - 3x - 3y = 0.$

Statement II Equation of a circle passing through the point, $(1, 1)$ and $(3, 3)$ is $x^2 + y^2 - 2x - 6y + 6 = 0$.

33 **Statement I** The circle of smallest radius passing through two given points A and B must be of radius $\frac{1}{2}AB$.

Statement II A straight line is a shortest distance between two points.

34 Consider $L_1 \equiv 2x + 3y + p - 3 = 0$,

$L_2 \equiv 2x + 3y + p + 3 = 0$, where p is a real number and $C \equiv x^2 + y^2 + 6x - 10y + 30 = 0$.

Statement I If line L_1 is a chord of circle C, then L_2 is not always a diameter of circle C.

Statement II If line L_1 is a diameter of circle C, then L_2 is not a chord of circle C.

35 **Statement I** The only circle having radius $\sqrt{10}$ and a diameter along line $2x + y = 5$ is

$x^2 + y^2 - 6x + 2y = 0.$

Statement II $2x + y = 5$ is a normal to the circle

$x^2 + y^2 - 6x + 2y = 0$ → **JEE Mains 2013**

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 If the line $y = x + 3$ meets the circle $x^2 + y^2 = a^2$ in A and B, then equation of the circle on AB as diameter is

(a) $x^2 + y^2 + 3x - 3y - a^2 + 9 = 0$
(b) $x^2 + y^2 - 3x + 3y - a^2 + 9 = 0$
(c) $x^2 + y^2 + 3x + 3y - a^2 + 9 = 0$
(d) None of the above

2 A circle C_1 of radius 2 units lies in the first quadrant and touches both the axes. Equation of the circle having centre at $(6, 5)$ and touching the circle C_1 externally is

(a) $x^2 + y^2 - 12x - 10y + 52 = 0$
(b) $x^2 + y^2 - 12x - 10y + 12 = 0$
(c) $x^2 + y^2 - 12x - 10y - 52 = 0$
(d) None of the above

3 The line $(x - 2)\cos\theta + (y - 2)\sin\theta = 1$ touches the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ for all values of θ, then $(g^2 + f^2 + c)/(g + f + c)$ is equal to

(a) 5 (b) 1
(c) –15 (d) None of these

4 Tangents PA and PB are drawn to $x^2 + y^2 = a^2$ from the point $P(x_1, y_1)$. Equation of the circumcircle of triangle PAB is

(a) $x^2 + y^2 - xx_1 - yy = 0$ (b) $x^2 + y^2 + xx_1 - yy_1 = 0$
(c) $x^2 + y^2 - xx_1 + yy_1 = 0$ (d) $x^2 + y^2 + xx_1 + yy_1 = 0$

5 If $a > 2b > 0$ then the positive value of m for which $y = mx - b\sqrt{(1 + m^2)}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$ is

(a) $\dfrac{2b}{\sqrt{a^2 - 4b^2}}$ (b) $\dfrac{\sqrt{a^2 - 4b^2}}{2b}$ (c) $\dfrac{2b}{1 - 2b}$ (d) $\dfrac{b}{a - 2b}$

6 The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \le 6$ into two parts.

If $S = \left\{\left(2, \frac{3}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}\right)\right\}$, then the number of point(s) in S lying inside the smaller part is

(a) 0 (b) 1 (c) 2 (d) 4

7 Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is

(a) 3 (b) 2 (c) $3/2$ (d) 1

8 If one of the diameters of the circle $S = x^2 + y^2 - 2x - 6y + 6 = 0$ is the common chord to the circle C with centre $(2, 1)$, then the radius of the circle is

(a) 1 (b) 3 (c) $\sqrt{3}$ (d) 2

9 A rational point is a point both of whose coordinates are rational numbers. Let C be any circle with centre $(0, \sqrt{2})$. Then, the maximum number of rational points on the circle is

(a) 0 (b) 2
(c) Infinitely many (d) None of these

10 Let L_1 be a line passing through the origin and L_2 be the line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then L_1 is

(a) $x + y = 0$ (b) $x + y = 2$
(c) $x + 7y = 0$ (d) $x - 7y = 0$

11 If a_n, $n = 1, 2, 3, 4$ represent four distinct positive real numbers other than unit such that each pair of the logarithm of a_n and the reciprocal of logarithm denotes a point on a circle, whose centre lies on Y-axis. Then, the product of these four members is

(a) 0 (b) 1 (c) 2 (d) 3

12 The set of values of a for which the point $(2a, a+1)$ is an interior point of the larger segment of the circle $x^2+y^2-2x-2y-8=0$ made by the chord $x-y+1=0$, is

(a) $\left(\frac{5}{9}, \frac{9}{5}\right)$ (b) $\left(0, \frac{5}{9}\right)$
(c) $\left(0, \frac{9}{5}\right)$ (d) $\left(1, \frac{9}{5}\right)$

13 Three concentric circles of which biggest circle is $x^2+y^2=1$, have their radii in AP. If the line $y=x+1$ cuts all the circles in real and distinct points, then the interval in which the common difference of AP will lie, is

(a) $\left[0, \left(1-\frac{1}{\sqrt{2}}\right)\right]$ (b) $\left(0, \frac{1}{2}\right)$
(c) $(1, 1)$ (d) $\left[0, \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)\right]$

14 If the distance from the origin of the centres of the three circles $x^2+y^2-2\lambda_i x=c^2$, $(i=1, 2, 3)$ are in G.P., then the lengths of the tangents drawn to them from any point on the circle $x^2+y^2=c^2$ are in

(a) AP (b) GP
(c) HP (d) None of these

15 The limiting points of the coaxial system of circles given by $x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+k)=0$ Subtend a tight angle at the origin, if

(a) $-\frac{c}{g^2}-\frac{k}{f^2}=2$ (b) $\frac{c}{g^2}+\frac{k}{f^2}=-2$
(c) $\frac{c}{g^2}-\frac{k}{f^2}=2$ (d) $\frac{c}{g^2}+\frac{k}{f^2}=2$

16 A ray of light incident at the point (3, 1) gets reflected from the tangent at $(0, 1)$ to the circle $x^2+y^2=1$. The reflected ray touches the circle. The equation of the line along which the incident ray moves is

(a) $3x+4y-13=0$ (b) $4x-3y-10=0$
(c) $4x+3y-13=0$ (d) $3x-4y-5=0$

17 The number of integral values of λ for which $x^2+y^2+\lambda x+(1-\lambda)y+5=0$ is the equation of a circle whose radius cannot exceed 5, is

(a) 14 (b) 18 (c) 16 (d) 10

18 The point $([P+1], [P])$ (where $[x]$ is the greatest integer less than or equal to x), lying inside the region bounded by the circles $x^2+y^2-2x-15=0$ and $x^2+y^2-2x-7=0$, then

(a) $P \in [-1, 2)-\{0, 1\}$ (b) $P \in [-1, 0) \cup [0, 1] \cup [1, 2)$
(c) $P \in (-1, 2)$ (d) None of these

19 The line $3x-4y-k=0$ touches the circle $x^2+y^2-4x-8y-5=0$ at (a, b). Then $k, (a, b)$ is

(a) 15, (5, 0) (b) –35, (–1, 8)
(c) Both (a) and (b) (d) None of these

20 If the equation of the circle obtained by reflecting the circle $x^2+y^2-a^2=0$ in the line $y=mx+c$ is $x^2+y^2+2gx+2fy+c=0$, then

(a) $g=\frac{2cm}{1+m^2}, a^2+c=\frac{4c^2}{1+m^2}$
(b) $g=-\frac{2cm}{1+m^2}, a^2+c=\frac{4c^2}{1+m^2}$
(c) $f=\frac{4c}{1+m^2}, a^2+c=\frac{4c^2}{1+m^2}$
(d) None of the above

ANSWERS

SESSION 1

1 (c)	**2** (a)	**3** (c)	**4** (a)	**5** (b)	**6** (a)	**7** (c)	**8** (b)	**9** (b)	**10** (c)
11 (d)	**12** (b)	**13** (a)	**14** (b)	**15** (a)	**16** (c)	**17** (c)	**18** (c)	**19** (a)	**20** (d)
21 (b)	**22** (d)	**23** (b)	**24** (c)	**25** (b)	**26** (c)	**27** (b)	**28** (a)	**29** (c)	**30** (c)
31 (a)	**32** (b)	**33** (b)	**34** (c)	**35** (c)					

SESSION 2

1 (a)	**2** (a)	**3** (a)	**4** (a)	**5** (a)	**6** (c)	**7** (b)	**8** (b)	**9** (b)	**10** (c)
11 (b)	**12** (c)	**13** (d)	**14** (b)	**15** (d)	**16** (a)	**17** (d)	**18** (d)	**19** (c)	**20** (a)

Hints and Explanations

SESSION I

1 Equation of two diameters are
$2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$
$\therefore$ Centre is $(-1/17, 22/17)$
Circle passes through origin
$\therefore$ Equation of the circle is
$$\left(x + \frac{1}{17}\right)^2 + \left(y - \frac{22}{17}\right)^2 = \left(\frac{1}{17}\right)^2 + \left(\frac{22}{17}\right)^2$$
$$\Rightarrow x^2 + y^2 + \frac{2}{17}x - \frac{44}{17}y = 0$$
$$\Rightarrow 17x^2 + 17y^2 + 2x - 44y = 0$$

2 Let circle be
$x^2 + y^2 + 2gx + 2fy + c = 0$
This passes through (2, 0), (0, 1), (4, 5)
So, $4 + 4g + c = 0$, $1 + 2f + c = 0$,
$41 + 8g + 10f + c = 0$
Solving these equations, we get
$g = -13/6$, $f = -17/6$, $c = 14/3$.
So, circle is
$3(x^2 + y^2) - 13x - 17y + 14 = 0$
Since $(0, a)$ also lies on it, we get
$\therefore \quad 3a^2 - 17a + 14 = 0$
So, $\quad a = 1$ or $14/3$

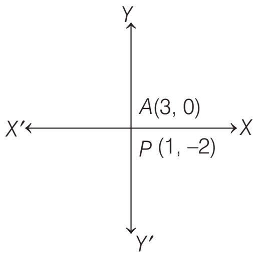
3 Let A and B be (x_1, y_1), (x_2, y_2).
x_1 and x_2 are roots of $x^2 + 2ax - 4 = 0$
Then, $x_1 + x_2 = -2a$, $x_1 x_2 = -4$.
y_1, y_2 are roots of $x^2 + 2bx - 9 = 0$.
Then, $y_1 + y_2 = -2b$, $y_1 y_2 = -9$.
Equations of circle on AB as diameter is
$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
$\Rightarrow x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1 x_2 + y_1 y_2 = 0$
$\Rightarrow \quad x^2 + y^2 + 2ax + 2by - 13 = 0$

4 The line $a_1 x + b_1 y + c_1 = 0$ cut X and Y-axis in $A(-c_1/a_1, 0)$ and $B(0, -c_1/b_1)$ and the line
$a_2 x + b_2 y + c_2 = 0$ cut axes in $C(-c_2/a_2, 0)$ and $D(0, -c_2/b_2)$,
So, AC and BD are chords along X and Y-axes intersecting at origin O.
Since A, B, C, D are concyclic so
$OA \cdot OC = OB \cdot OD$
or $\left(-\frac{c_1}{a_1}\right)\left(-\frac{c_2}{a_2}\right) = \left(-\frac{c_1}{b_1}\right)\left(\frac{c_2}{b_2}\right)$
$\Rightarrow \quad a_1 a_2 = b_1 b_2$.

5 Given lines are $3x + 5y = 1$ $\quad$ ...(i)
and $\quad (2 + c)x + 5c^2 y = 1$ $\quad$ ...(ii)
From Eqs. (i) and (ii), we get
$(1 - c)x + 5(1 - c^2)y = 0$
$c = 1, x + 10y = 0$
$\therefore$ Centre $= \left(\frac{2}{5}, -\frac{1}{25}\right)$
$\therefore$ Equation of circle is
$$\left(x - \frac{2}{5}\right)^2 + \left(y + \frac{1}{25}\right)^2 = \left(2 - \frac{2}{5}\right)^2 + \frac{1}{25^2}$$
$\Rightarrow 25(x^2 + y^2) - 20x + 2y - 60 = 0$

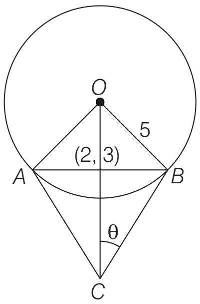
6 Here, $3 \times 1 = (-1)(-3) = 3$
Hence, the points are concyclic.
$\therefore \quad L_1 L_2 + \lambda xy = 0$
$\Rightarrow (3x - y + 3)(x - 3y - 6) + \lambda\, xy = 0$
$3(x^2 + y^2) + (\lambda - 10)xy - 15x - 3y - 18 = 0$
Now for circle coefficient of $xy = 0$
$\lambda - 10 = 0 \Rightarrow \lambda = 10$
$\therefore \quad 3(x^2 + y^2) - 15x - 3y - 18 = 0$
$\Rightarrow \quad x^2 + y^2 - 5x - y - 6 = 0$

7 Let the equation of circle be
$(x - 3)^2 + (y - 0)^2 + \lambda y = 0$



As it passes through $(1, -2)$.
$\therefore \quad (1 - 3)^2 + (-2)^2 + \lambda(-2) = 0$
$\Rightarrow \quad 4 + 4 - 2\lambda = 0$
$\Rightarrow \quad \lambda = 4$
$\therefore$ Equation of circle is
$(x - 3)^2 + y^2 + 4y = 0$
Now, by hit and trial method, we see that point $(5, -2)$ satisfies equation of circle.

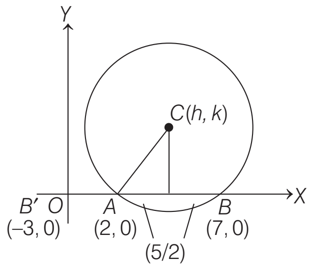
8 Area of quadrilateral $OACB$,



$A = OB \cdot BC = 5^2 \cot\theta$
$= 50\sqrt{\frac{3}{13}} \quad \left[\because \sin\theta = \frac{\sqrt{13}}{5}\right]$

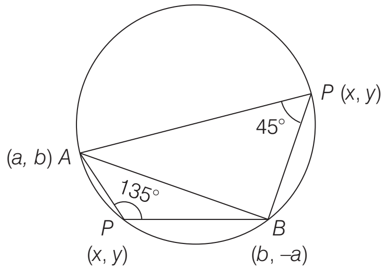
9 Length of side of triangle is 2.
The equation of circle passing through $(-1, 0)$ and $(1, 0)$ is
$(x + 1)(x - 1) + y^2 + y\lambda = 0$.
The coordinates of third vertex will be $(0, \sqrt{3})$, which is passing by the circle.
$\therefore \quad \lambda = -\frac{2}{\sqrt{3}}$
So, the equation of circle is
$x^2 + y^2 - \frac{2y}{\sqrt{3}} - 1 = 0$.

10 Here, clearly, x-coordinate of centre is x- coordinate of the mid-point of AB or AB' i.e. $9/2$ or $-1/2$.



Since centre lies in the first quadrant,
$h = 9/2$
$\therefore$ Equation of circle is
$(x - 9/2)^2 + (y - k)^2 = (h - 2)^2 + k^2 = 25/4 + k^2$
or $\quad x^2 + y^2 - 9x - 2ky + 14 = 0$

11 Let $P(x, y)$ be any point on the circumference of the circle.
Then, m_1 = Slope of $PA = \frac{b - y}{a - x}$
and $\quad m_2$ = Slope of $PB = \frac{-a - y}{b - x}$



We have, $\angle APB = 45°$ or $135°$
$$\Rightarrow \quad \frac{m_1 - m_2}{1 + m_1 m_2} = \tan 45° \text{ or } \tan 135°$$
$$\Rightarrow \quad \frac{\frac{b - y}{a - x} - \frac{-a - y}{b - x}}{1 + \frac{b - y}{a - x} \times \frac{-a - y}{b - x}} = \pm 1$$

$$\Rightarrow \frac{(b-y)(b-x)+(a+y)(a-x)}{(a-x)(b-x)-(b-y)(a+y)} = \pm 1$$

$$\Rightarrow \quad x^2 + y^2 = a^2 + b^2$$

$$\Rightarrow \quad \{x-(a+b)\}^2 + \{y-(b-a)\}^2 = a^2 + b^2$$

The centres of these circles are $O(0,0)$ and $C(a+b, b-a)$.

$\therefore$ Distance between the centre

$$= \sqrt{(a+b)^2 + (a-b)^2} = \sqrt{2}\sqrt{a^2+b^2}$$

$= \sqrt{2}$ (Radius of either circle)

12 The tangent at $B(1, 7)$ is $y = 7$ and $D(4, -2)$ is $3x - 4y - 20 = 0$.

Then, meet at $C(16, 7)$.

Now, $AB = 5$, $BC = 15$

Area of quadrilateral $ABCD$

$$= AB \cdot BC = 75$$

13 Centre of given circle

$x^2 + y^2 - 6x + 12y + 15 = 0$ is $(3, -6)$.

$\therefore$ Radius $= \sqrt{(3)^2 + (-6)^2 - 15} = \sqrt{30}$

Area of circle $= \pi r^2 = \pi(\sqrt{30})^2 = 30\pi$

Area of required circle

$= 2$ (Area of given circle)

$\therefore \quad \pi R^2 = 2 \times 30\pi = 60\pi$

$\Rightarrow \quad R^2 = 60 \Rightarrow R = 2\sqrt{15}$

$\therefore$ Equation of required circle is

$$(x-3)^2 + (y+6)^2 = (2\sqrt{15})^2$$

$\Rightarrow x^2 + 9 - 6x + y^2 + 36 + 12y = 60$

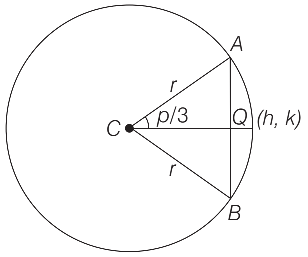
$\Rightarrow \quad x^2 + y^2 - 6x + 12y - 15 = 0$

14 Let $Q(h, k)$ be the mid-point of chord AB.

$\therefore \quad \angle ACQ = \angle BCQ = \pi/3$

Coordinate of centre are

$(3/2, -1/2)$ and radius $= 3/2$



Now, $CQ = r\cos\pi/3 = r/2 = 3/4$.

$\Rightarrow \quad (h - 3/2)^2 + (k + 1/2)^2 = (3/4)^2$

$\therefore \quad$ Locus of $Q(h, k)$ is

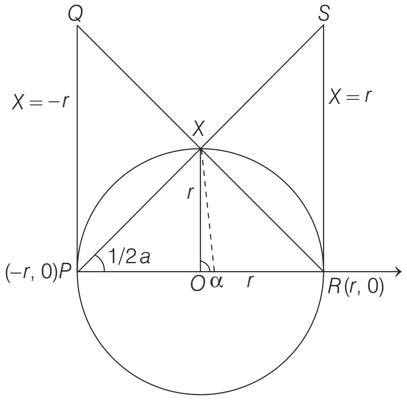
$$x^2 + y^2 - 3x + y + 31/16 = 0.$$

15 Taking diameter PR as X-axis and centre O as origin, tangents at P and R are given by

$$x = -r \quad \ldots(i)$$

and $$x = r \quad \ldots(ii)$$

Let coordinates of X on circle be $(r\cos\alpha, r\sin\alpha)$



$$\angle XPR = \frac{1}{2}\angle XOR = \frac{1}{2}\alpha$$

$\therefore \quad RS = 2r\tan\frac{1}{2}\alpha;$

$\angle XRP = 90^\circ - \alpha/2$

$\therefore \quad PQ = 2r\tan(90^\circ - \alpha/2)$

$= 2r\cot\alpha/2$

$\therefore \quad PQ.RS = 4r^2 \Rightarrow 2r = \sqrt{(PQ.RS)}$

16 Line $ax + by = 0$ is normal to the circle $x^2 + y^2 - 4x + 2y - 3 = 0$, so its centre $(2, -1)$ lie on line.

$\therefore \quad 2a - b = 0$ *i.e.*, $b = 2a \quad \ldots(i)$

Also line touches the circle

$x^2 + y^2 + 2x + 4y = 0$

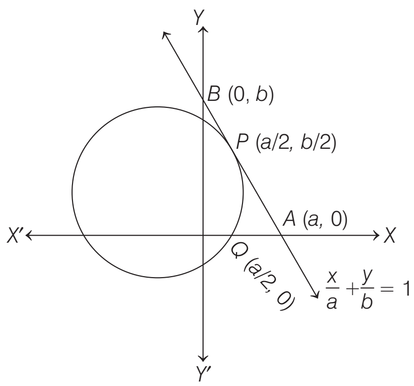
$$\therefore \quad \left|\frac{-a-2b}{\sqrt{(a^2+b^2)}}\right| = \sqrt{5} \quad \ldots(ii)$$

Solving (i) and (ii), we get $4a^2 = b^2$.

From the choices, only solution is $(1, 2)$

17 The equation of the circle touching

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ at } P\left(\frac{a}{2}, \frac{b}{2}\right) \text{ is } \left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 + \lambda\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0 \quad \ldots(i)$$



Since, it passes through $Q\left(\frac{a}{2}, 0\right)$.

Therefore, $\lambda = \frac{b^2}{2}$ [from Eq. (i)]

On putting the value of λ in Eq. (i), we get

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 + \frac{b^2}{2}\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0$$

$$\Rightarrow \quad x^2 + y^2 - \left(a - \frac{b^2}{2a}\right)x - \frac{by}{2} + \frac{a^2 - b^2}{4} = 0$$

Let r be the radius of this circle. Then,

$$r^2 = \frac{1}{4}\left(a - \frac{b^2}{2a}\right)^2 + \frac{b^2}{16} - \left(\frac{a^2 - b^2}{4}\right)$$

$$= \frac{b^2(a^2 + b^2)}{16a^2} \Rightarrow r = \frac{b}{4a}\sqrt{a^2 + b^2}$$

18 Given equations of circles are

$$x^2 + y^2 - 4x - 6y - 12 = 0 \quad \ldots(i)$$

$$x^2 + y^2 + 6x + 18y + 26 = 0 \quad \ldots(ii)$$

Centre of circle (i) is $C_1(2, 3)$ and radius

$= \sqrt{4 + 9 + 12} = 5(r_1)$ [say]

Centre of circle (ii) is $C_2(-3, -9)$ and radius

$= \sqrt{9 + 81 - 26} = 8(r_2)$ [say]

Now, $C_1C_2 = \sqrt{(2+3)^2 + (3+9)^2}$

$\Rightarrow \quad C_1C_2 = \sqrt{5^2 + 12^2}$

$\Rightarrow \quad C_1C_2 = \sqrt{25 + 144} = 13$

Also, $r_1 + r_2 = 5 + 8 = 13$

$\therefore \quad C_1C_2 = r_1 + r_2$

Thus, both circles touch each other externally. Hence, there are three common tangents.

19 Centres and radii of given circles are

$C_1(1, 3)$, $r_1 = r$, $C_2(4, -1)$

and $r_2 = \sqrt{4^2 + 1^2 - 8} = 3$

Since, $r_1 - r_2 < C_1C_2 < r_1 + r_2$

$\Rightarrow r - 3 < \sqrt{(4-1)^2 + (-1-3)^2} < r + 3$

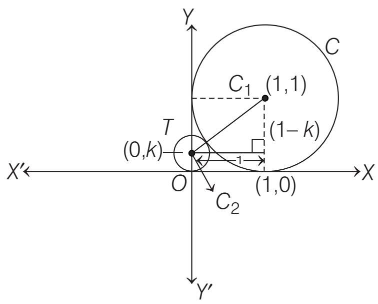
$\Rightarrow \quad r - 3 < 5 < r + 3$

$\Rightarrow \quad r - 3 < 5$ and $5 < r + 3$

$\Rightarrow \quad r < 8$ and $2 < r \quad \Rightarrow \quad 2 < r < 8$

20 Use the property, when two circles touch each other externally, then distance between the centre is equal to sum of their radii, to get required radius.

Let the coordinate of the centre of T be $(0, k)$.



Distance between their centre

$$k+1=\sqrt{1+(k-1)^2}$$

$[\because C_1C_2 = k+1]$

$\Rightarrow\ k+1=\sqrt{1+k^2+1-2k}$

$\Rightarrow\ k+1=\sqrt{k^2+2-2k}$

$\Rightarrow\ k^2+1+2k=k^2+2-2k$

$\Rightarrow\ k=\dfrac{1}{4}$

So, the radius of circle T is k i.e. $\dfrac{1}{4}$.

21 Equation of tangent at $(1,-2)$ is $x-2y-5=0$. For IInd circle centre $(4,-3)$ and $r=\sqrt{5}$.

Point of contact is $\dfrac{x-4}{1}=\dfrac{y+3}{-2}$

$$=-\frac{(4+6-5)}{1^2+2^2}=-1$$

$\Rightarrow\ \dfrac{x-4}{1}=-1,\ \dfrac{y+3}{-2}=-1$

$\therefore\ x=3,\ y=-1$

22 Equation of tangent at $(1,7)$ to the curve $x^2=y-6$ is

$$x=\left(\frac{y+7}{2}\right)-6$$

$\Rightarrow\ 2x-y+5=0$

Since, $2x-y+5=0$ touches the circle

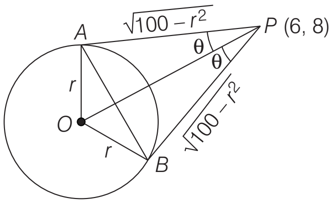
$$x^2+y^2+16x+12y=C=0$$

$\therefore\ \sqrt{(8)^2+(6)^2-c}=\left|\dfrac{2(-8)-(-6)+5}{\sqrt{(2)^2+(1)^2}}\right|$

$\Rightarrow\ \sqrt{100-c}=\sqrt{5}$

$\Rightarrow\ c=95$

23 Now, $OP=\sqrt{6^2+8^2}=10$



$$PA=\sqrt{S_1}=\sqrt{100-r^2}$$

Let $f(r)=\Delta PAB=\dfrac{1}{2}PA\cdot PB\cdot\sin 2\theta$

$$=(100-r^2)\sin\theta\cdot\cos\theta$$

$$=\frac{r}{100}(100-r^2)^{3/2}$$

Put $f'(r)=0$

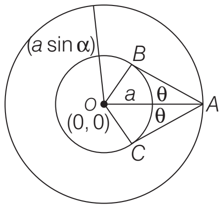
$\Rightarrow\ \dfrac{3}{2}(100-r^2)^{1/2}(-2r^2)+(100-r^2)^{3/2}$

$\Rightarrow\ \sqrt{100-r^2}\,(-3r^2+100-r^2)=0$

$\Rightarrow\ r=\pm 10$ or $r=5$

Hence, $r=5$

24 Let angles between the tangents $=2\theta$, then



$\sin\theta=OB/OA=(a\sin\alpha/a)=\sin\alpha$

So, $\theta=\alpha$.

Required angle $=2\theta=2\alpha$

25 We have, $x^2+y^2-4x+6y-12=0$

Centre $(2,-3)$

Radius $=\sqrt{(2)^2+(-3)^2+12}$

$$=\sqrt{4+9+12}=5$$

Distance between two centres $c_1(2,-3)$ and $c_2(-3,2)$

$$d=\sqrt{(2+3)^2+(-3-2)^2}$$

$$=\sqrt{25+25}=\sqrt{50}$$

Radius of circle $S=\sqrt{5^2+(\sqrt{50})^2}$

$$=\sqrt{25+50}=5\sqrt{3}$$

26 The equations of two circles are

$S_1\equiv(x-a)^2+(y-b)^2=c^2$ ...(i)

and $S_2\equiv(x-b)^2+(y-a)^2=c^2$ ...(ii)

The equation of the common chord of these circles is

$$S_1-S_2=0$$

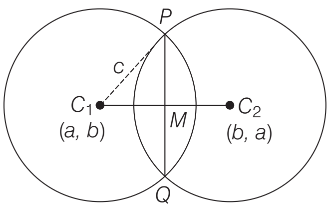
$\Rightarrow\ (x-a)^2-(x-b)^2+(y-b)^2-(y-a)^2=0$

$\Rightarrow\ (2x-a-b)(b-a)+(2y-b-a)(a-b)=0$

$\Rightarrow\ 2x-a-b-2y+b+a=0$

$\Rightarrow\ x-y=0$

The centre coordinates of circles S_1 and S_2 are $C_1(a,b)$ and $C_2(b,a)$, respectively.



Now, $C_1M=\left|\dfrac{a-b}{\sqrt{1+1}}\right|=\dfrac{|a-b|}{\sqrt{2}}$

In right ΔC_1PM,

$$PM=\sqrt{C_1P^2-C_1M^2}=\sqrt{c^2-\frac{(a-b)^2}{2}}$$

$\therefore\ PQ=2PM=2\sqrt{c^2-\dfrac{(a-b)^2}{2}}$

$$=\sqrt{4c^2-2(a-b)^2}$$

27 Let equation of circle be

$$x^2+y^2+2gx+c=0.$$

where, g is a variable and c is a constant, be a coaxial system of circle having common radical axis as X-axis.

Let $x^2+y^2+2g_i\,x+c=0;\ i=1,2,3$ be three members of the given coaxial system of circles.

Then, the coordinates of their centres and radii are

$$P(-g_1,0),\ Q(-g_2,0),\ R(-g_3,0)$$

and $r_1^2=g_1^2-c,\ r_2^2=g_2^2-c,\ r_3^2=g_3^2-c$

Now, $r_1^2\cdot QR+r_2^2\cdot RP+r_3^2\cdot PQ$

$=(g_1^2-c)(g_2-g_3)+(g_2^2-c)(g_3-g_1)+(g_3^2-c)(g_1-g_2)$

$=g_1^2(g_2-g_3)+g_2^2(g_3-g_1)+g_3^2(g_1-g_2)-c\{(g_2-g_3)+(g_3-g_1)+(g_1-g_2)\}$

$=-(g_1-g_2)\cdot(g_2-g_3)\cdot(g_3-g_1)$

$=-PQ\cdot QR\cdot RP$

28 The combined equation of the lines joining the origin to the points of intersection of $x\cos\alpha+y\sin\alpha=p$ and $x^2+y^2-a^2=0$ is a homogeneous equation of second degree in given by

$$x^2+y^2-a^2\left(\frac{x\cos\alpha+y\sin\alpha}{p}\right)^2=0$$

$\Rightarrow\ x^2(p^2-a^2\cos^2\alpha)+y^2(p^2-a^2\sin^2\alpha)-2xya^2\sin\alpha\cos\alpha=0$

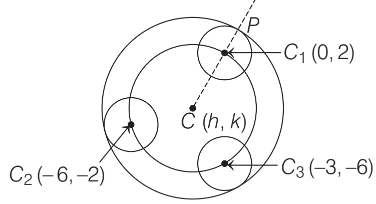
The lines given by this equation are at right angle if

$(p^2-a^2\cos^2\alpha)+(p^2-a^2\sin^2\alpha)=0$

$\Rightarrow\ 2p^2=a^2(\sin^2\alpha+\cos^2\alpha)$

$\Rightarrow\ a^2=2p^2$

29 The coordinates of the centres and radii of three given circles are as given



Circle	Centre	Radius
Ist Circle	$C_1(0,2)$	$r_1=3$
IInd Circle	$C_2(-6,-2)$	$r_2=3$
IIIrd Circle	$C_3(-3,-6)$	$r_3=3$

Let $C(h,k)$ be the centre of the circle passing through the centres of the circles Ist, IInd and IIIrd.

Then, $CC_1=CC_2=CC_3$

$\Rightarrow\ CC_1^2=CC_2^2=CC_3^2$

$\Rightarrow (h-0)^2+(k-2)^2=(h+6)^2+(k+2)^2=(h+3)^2+(k+6)^2$

$\Rightarrow \quad -4k + 4 = 12h + 4k + 40$
$\quad = 6h + 12k + 45$

$\Rightarrow \quad 12h + 8k + 36 = 0$

and $\quad 6h - 8k - 5 = 0$

$\Rightarrow \quad 3h + 2k + 9 = 0$

and $\quad 6h - 8k - 5 = 0$

$\Rightarrow \quad h = \frac{-31}{18}, k = \frac{-23}{12}$

$\therefore \quad CC_1 = \sqrt{\left(0 + \frac{31}{18}\right)^2 + \left(2 + \frac{23}{12}\right)^2}$

$= \frac{5}{36}\sqrt{949}$

Now, $CP = CC_1 + C_1P$

$\Rightarrow \quad CP = \left(\frac{5}{36}\sqrt{949} + 3\right)$

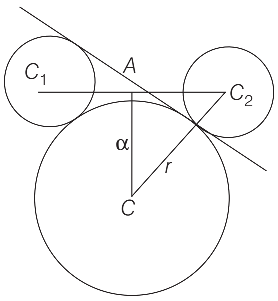
Thus, required circle has its centre at $\left(\frac{-31}{18}, \frac{-23}{12}\right)$ and radius

$= CP = \left(\frac{5}{36}\sqrt{949} + 3\right)$.

Hence, its equation is

$$\left(x + \frac{31}{18}\right)^2 + \left(y + \frac{23}{12}\right)^2 = \left(3 + \frac{5}{36}\sqrt{949}\right)^2$$

30 $(r+1)^2 = \alpha^2 + 9; \; r^2 + 8 = \alpha^2$



$\Rightarrow \quad r^2 + 2r + 1 = r^2 + 8 + 9$
$\quad 2r = 16 \Rightarrow r = 8$

31 Since, the tangents are perpendicular. So, locus of perpendicular tangents to the circle $x^2 + y^2 = 169$ is a director circle having equation $x^2 + y^2 = 338$.

32 Equation of circle, when the limiting points are (1, 1) and (3, 3) is

$(x-1)^2 + (y-1)^2 = 0$

and $\quad (x-3)^2 + (y-3)^2 = 0$

$\Rightarrow \quad x^2 + y^2 - 2x - 2y + 2 = 0$

and $\quad x^2 + y^2 - 6x - 6y + 18 = 0$

Equation of the coaxial system of circle is

$x^2 + y^2 - 6x - 6y + 18$
$\quad + \lambda(x^2 + y^2 - 2x - 2y + 2) = 0$

It passes through origin, therefore

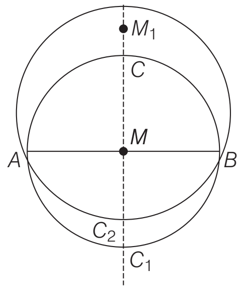
$\lambda = -9$

Hence, required circle is

$2x^2 + 2y^2 - 3x - 3y = 0.$

Statement II is also true but it is not a correct explanation for Statement I.

33 Let C_1 be a circle which passes through A, B and C whose diameter is AB and C_2 be another circle which passes through A and B, then centres of C_1 and C_2 must lie on perpendicular bisector of AB. Indeed centre of C_1 is mid-point M of AB and centre of any other circle lies somewhere else on bisector.



Then, $\quad M_1A > AM$
[hypotenuse of right angled ΔAMM_1]

$\Rightarrow \quad$ Radius of $C_2 > \frac{1}{2}AB$

So, C_1 is the circle whose radius is least.

Thus, Statement I is true but does not actually follow from Statement II which is certainly true.

34 Equation of circle C is

$(x+3)^2 + (y-5)^2 = 9 + 25 - 30$

i.e. $(x+3)^2 + (y-5)^2 = 4$

$\therefore \quad$ Centre is $(3, -5)$.

If L_1 is a diameter of a circle, then

$2(3) + 3(-5) + p - 3 = 0$

$\Rightarrow \quad p = 12$

$\therefore \quad L_1$ is $2x + 3y + 9 = 0$

and $\quad L_2$ is $2x + 3y + 15 = 0$.

Distance of centre of circle C from L_2

$= \left|\frac{2(3) + 3(-5) + 15}{\sqrt{2^2 + 3^2}}\right|$

$= \frac{6}{\sqrt{13}} < 2$

Hence, L_2 is a chord of circle C.

35 **Statement I** Centre of circle $= (3, -1)$

Now, $2(3) + (-1) = 5 = 5$, [true]

Statement II Centre $= (3, -1)$, which lies on given line.

Simplify it and get the result.

SESSION 2

1 Equation of circle through points of intersections of circle $x^2 + y^2 - a^2 = 0$ and line $x - y + 3 = 0$, (AB) is

$(x^2 + y^2 - a^2) + \lambda(x - y + 3) = 0$

Since, AB is diameter, centre $\left(-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ lies on it.

$\therefore \quad -\frac{\lambda}{2} - \frac{\lambda}{2} + 3 = 0$ i.e. $\lambda = 3$

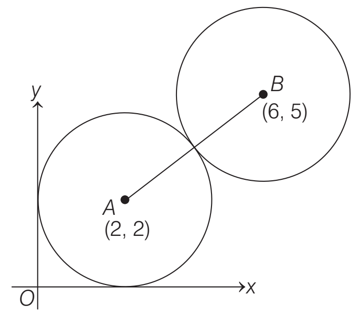
Hence, equation of required circle is

$x^2 + y^2 + 3x - 3y + 9 - a^2 = 0$

2 $AB = \sqrt{(6-2)^2 + (5-2)^2} = 5$

$AC = 2$

$\Rightarrow \quad BC = 3$



Equation of required circle is

$(x-6)^2 + (y-5)^2 = 9.$

$\Rightarrow \quad x^2 + 36 - 12x + y^2 + 25 - 10y = 9$

$\Rightarrow \quad x^2 + y^2 - 12x - 10y + 52 = 0$

3 The line $(x-2)\cos\theta + (y-2)\sin\theta = 1$ touches the circle

$(x-2)^2 + (y-2)^2 = 1$

i.e. $\quad x^2 + y^2 - 4x - 4y + 7 = 0$

for all values of θ.

Comparing it with the circle

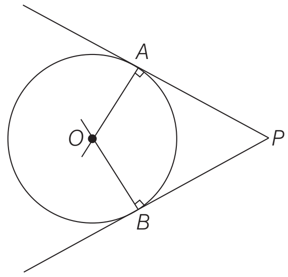
$x^2 + y^2 + 2gx + 2fy + c = 0$

we get, $\quad g = -2, f = -2, c = 7$

$\therefore \quad \frac{g^2 + f^2 + c}{g + f + c} = \frac{4 + 4 + 7}{-2 - 2 + 7} = \frac{15}{3} = 5$

4 $\angle PAO = \angle PBO = \pi/2.$

$\therefore \quad P, A, O, B$ are concyclic.



$\therefore \quad$ Equation of circumcircle of ΔABP is

$x(x - x_1) + y(y - y_1) = 0$

i.e. $\quad x^2 + y^2 - xx_1 - yy_1 = 0$

5 The line $y = mx - b\sqrt{1+m^2}$ ...(i)

is tangent to the circle $x^2 + y^2 = b^2$

It is also tangent to the circle

$$(x-a)^2 + y^2 = b^2 \quad ...(ii)$$

It is length of $\perp$ on (i) from centre $(a, 0)$

= radius of circle (ii)

$$\Rightarrow \left|\frac{ma - b\sqrt{(1+m^2)}}{\sqrt{1+m^2}}\right| = b$$

$$\Rightarrow ma - b\sqrt{(1+m^2)} = \pm b\sqrt{(1+m^2)} \quad ...(iii)$$

– ve sign givess $m = 0$, which is none of the given options.

$\therefore$ Taking + ve sign on RHS of (iii), we get

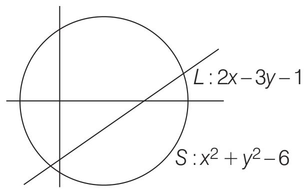
$$ma = 2b\sqrt{(1+m^2)}$$

$$\Rightarrow m = 2b/\sqrt{(a^2 - 4b^2)}.$$

6 $L : 2x - 3y - 1; S : x^2 + y^2 - 6$

If $L_1 > 0$ and $S_1 < 0$

Then, point lies in the smaller part are as

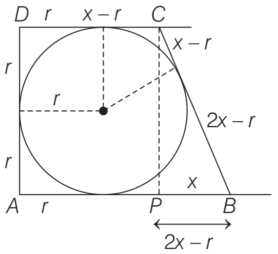


$\therefore \left(2, \frac{3}{4}\right)$ and $\left(\frac{1}{4}, -\frac{1}{4}\right)$ lie inside.

7 $\frac{1}{2}(x + 2x)2r = 18$

$(3x - 2r)^2 = 4r^2 + x^2$

On solving for r, we get $r = 2$



8 $S = x^2 + y^2 - 2x - 6y + 6 = 0,$

Centre = (1, 3)

Let radius of circle $C = r$.

Then, $C = (x-2)^2 + (y-1)^2 = r^2$

$= x^2 + y^2 - 4x - 2y + 5 - r^2 = 0$

Common chord of circles S and C is

$$2x - 4y + 1 - r^2 = 0$$

It is a diameter of circle S.

$\therefore \quad 2 - 12 + 1 - r^2 = 0 \Rightarrow r = 3$

9 $S = x^2 + y^2 - 2\sqrt{2}y + c = 0$

Let us assume that there are more than two rational points on the circle. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ $x_i, y_i \in Q, i = 1, 2, 3$ be three rational points.

Then, $x_1^2 + y_1^2 - 2\sqrt{2}y_1 = x_2^2 + y_2^2 - 2\sqrt{2}y_2 = x_3^2 + y_3^2 - 2\sqrt{2}y_3$

$\Rightarrow \quad x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2$

and $\quad y_1 = y_2 = y_3 \Rightarrow x_1^2 = x_2^2 = x_3^3$

$\Rightarrow$ There exists two rational points of the form (a, b) and $(-a, b)$, $a, b \in Q$.

10 The chords are equal length, then the distances of the centre from the lines are equal. Let L_1 be $y - mx = 0$ centre is $\left(\frac{1}{2}, -\frac{3}{2}\right)$.

$$\therefore \frac{\left|-\frac{3}{2} - \frac{m}{2}\right|}{\sqrt{m^2+1}} = \frac{\left|\frac{1}{2} - \frac{3}{2} - 1\right|}{\sqrt{2}}$$

$$\Rightarrow 7m^2 - 6m - 1 = 0$$

$$\Rightarrow m = 1, -\frac{1}{7}$$

Hence, L_1 be $y + \frac{1}{7}x = 0 \Rightarrow x + 7y = 0$

11 Let $(0, b)$ be the centre and r be the radius of the given circle, then its equation is

$$(x-0)^2 + (y-b)^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2yb + b^2 - r^2 = 0 \quad ...(i)$$

It is given that the point

$P_n\left(\log a_n, \frac{1}{\log a_n}\right)$; $n = 1,2,3,4$ lie on the circle given by Eq. (i).

Therefore, $(\log a_n)^2 + \frac{1}{(\log a_n)^2} - \frac{2b}{(\log a_n)}$, $+ b^2 - r^2 = 0$

$n = 1,2,3,4$

Since, $\log a_1$, $\log a_2$, $\log a_3$ and $\log a_4$ are roots of the equation.

Then, $\lambda^4 + (b^2 - r^2)\lambda^2 - 2b\lambda + 1 = 0$

$\therefore$ Sum of the roots = 0

$\Rightarrow \log a_1 + \log a_2 + \log a_3 + \log a_4 = 0$

$\Rightarrow \log(a_1\, a_2\, a_3\, a_4) = 0 \Rightarrow a_1 a_2 a_3 a_4 = 1$

12 The point $(2a, a+1)$ will be an interior point of the larger segment of the circle $x^2 + y^2 - 2x - 2y - 8 = 0$

(i) The point $(2a, a+1)$ is an interior point.

(ii) The point $(2a, a+1)$ and the centre $(1,1)$ are on the same side of the chord $x - y + 1 = 0$.

$$\therefore \quad (2a)^2 + (a+1)^2 - 2(2a) - 2(a+1) - 8 < 0$$

and $(2a - a - 1 + 1)(1 - 1 + 1) > 0$

$\Rightarrow \quad 5a^2 - 4a - 9 < 0$ and $a > 0$

$\Rightarrow \quad (5a - 9)(a+1) < 0$ and $a > 0$

$\Rightarrow \quad -1 < a < \frac{9}{5}$ and $a > 0$

$\Rightarrow \quad a \in \left(0, \frac{9}{5}\right)$

13 The equation of the biggest circle is $x^2 + y^2 = 1^2$

Clearly, it is centred at $O(0, 0)$ and has radius 1. Let the radii of the other two circles be $1 - r$, $1 - 2r$, where $r > 0$.

Thus, the equations of the concentric circles are

$$x^2 + y^2 = 1 \quad ...(i)$$

$$x^2 + y^2 = (1-r)^2 \quad ...(ii)$$

$$x^2 + y^2 = (1-2r)^2 \quad ...(iii)$$

Clearly, $y = x + 1$ cuts the circle (i) at (1, 0) and (0, 1). This line will cut circles (ii) and (iii) in real and distinct points, if

$$\left|\frac{1}{\sqrt{2}}\right| < 1 - r \text{ and } \left|\frac{1}{\sqrt{2}}\right| < 1 - 2r$$

$$\Rightarrow \frac{1}{\sqrt{2}} < 1 - r \text{ and } \frac{1}{\sqrt{2}} < 1 - 2r$$

$$\Rightarrow r < 1 - \frac{1}{\sqrt{2}} \text{ and } r < \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow r < \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow r \in \left[0, \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\right] \quad [\because r > 0]$$

14 Given circles are

$$x^2 + y^2 - 2\lambda_i x - c^2 = 0,\ i = 1, 2, 3$$

$\therefore$ Centres are $(\lambda_1, 0), (\lambda_2, 0), (\lambda_3, 0)$

Distances of the centres from origin are in G.P.,

$\therefore \lambda_2^2 = \lambda_1 \lambda_3$

Let (x_1, y_1) be any point on the circle $x^2 + y^2 = c^2$.

$\therefore \quad x_1^2 + y_1^2 - c^2 = 0$

Lengths of tangents from (x_1, y_1) to the three given circles are

$$l_i = \sqrt{x_1^2 + y_1^2 - 2\lambda_j x_1 - c^2} = \sqrt{-2\lambda_j x_1}$$

$$\therefore \quad l_1^2\, l_3^2 = (-2\lambda_1 x_1)(-2\lambda_3 x_1)$$

$$= 4\lambda_1\lambda_3 x_1^2 = 4\lambda_2^2 x_1^2 = \lambda_2^4$$

$\therefore \quad l_1, l_2$ and l_3 are in G.P.

15 The equation representing the coaxial system of circle is

$$x^2 + y^2 + 2gx + c + \lambda(x^2 + y^2 + 2fy + k) = 0$$

$$\Rightarrow x^2 + y^2 + \frac{2g}{1+\lambda}x + \frac{2f\lambda}{1+\lambda}y + \frac{c + k\lambda}{1+\lambda} = 0 \quad ...(i)$$

The coordinates of the centre of this circle are

$$\left(-\frac{g}{1+\lambda}, -\frac{f\lambda}{1+\lambda}\right) \quad ...(ii)$$

and radius

$$= \sqrt{\frac{g^2 + f^2\lambda^2 - (c + k\lambda)(1 + \lambda)}{(1 + \lambda)^2}}$$

For the limiting points,
we must have

Radius = 0

$\Rightarrow \quad g^2 + f^2\lambda^2 - (c + k\lambda)(1 + \lambda) = 0$

$\Rightarrow \quad \lambda^2(f^2 - k) - \lambda(c + k) + (g^2 - c) = 0 \quad \ldots$(iii)

Let λ_1 and λ_2 be the roots of this equation.

Then, $\quad \lambda_1 + \lambda_2 = \dfrac{c + k}{f^2 - k}$

and $\quad \lambda_1\lambda_2 = \dfrac{g^2 - c}{f^2 - k} \quad \ldots$(iv)

Thus, the coordinates of limiting points L_1 and L_2 are,

$L_1\left(\dfrac{-g}{1 + \lambda_1}, \dfrac{-f\lambda_1}{1 + \lambda_1}\right)$ and

$L_2\left(\dfrac{-g}{1 + \lambda_2}, \dfrac{-f\lambda_2}{1 + \lambda_2}\right)$ [from Eq. (iv)]

Now, $L_1 L_2$ will subtend a right angle at the origin.

If slope of $OL_1 \times$ slope of $OL_2 = -1$

$\Rightarrow \quad \dfrac{f\lambda_1}{g} \times \dfrac{f\lambda_2}{g} = -1$

$\Rightarrow \quad f^2\lambda_1\lambda_2 = -g^2$

$\Rightarrow \quad f^2\left(\dfrac{g^2 - c}{f^2 - k}\right) = -g^2$

$\Rightarrow \quad f^2(g^2 - c) + g^2(f^2 - k) = 0$

$\Rightarrow \quad \dfrac{c}{g^2} + \dfrac{k}{f^2} = 2$

16 Tangent at (0, 1) to the circle $x^2 + y^2 = 1$ is $y = 1$. Incident ray, incident at (3, 1) is $y - 1 = m(x - 3)$
Incident and reflected rays are equally and reflected rays are equally inclined to the line is $y = 1$, so slope of reflected ray is $-m$.

$\therefore$ Equation of reflected ray is

$y - 1 = -m(x - 3)$

i.e. $\quad mx + y - 3m - 1 = 0$

It touches the circle so $(p = r)$

$$\left|\frac{3m + 1}{\sqrt{1 + m^2}}\right| = 1$$

$\Rightarrow \quad 9m^2 + 6m + 1 = 1 + m^2$

$\Rightarrow \quad m = 0, -3/4.$

$m = 0$ gives the slope for the tangent $y = 1$, so equation of reflected ray is

$y - 1 = -\dfrac{3}{4}(x - 3)$

i.e., $3x + 4y - 13 = 0$

17 $S = x^2 + \lambda x + (1 - \lambda)y + 5 = 0$

is a circle of radius not exceeding 5.

$\therefore \quad \sqrt{\dfrac{\lambda^2}{4} + \dfrac{(1 - \lambda)^2}{4} - 5} \le 5$

and $\quad \dfrac{\lambda^2}{4} + \dfrac{(1 - \lambda)^2}{4} - 5 \ge 0$

$\therefore \quad \lambda^2 + (1 - \lambda)^2 - 20 \le 100$

and $\quad \lambda^2 + (1 - \lambda)^2 - 20 > 0$

$\Rightarrow \quad \dfrac{1 - \sqrt{239}}{2} \le \lambda \le \dfrac{1 + \sqrt{239}}{2}$

i.e. $-7.2 \le \lambda \le 8.2$

and $\lambda < \dfrac{1 - \sqrt{39}}{2}$ or $\lambda > \dfrac{1 + \sqrt{39}}{2}$

i.e. $\quad \lambda - 2.62$ or $\lambda > 3.62$

$\therefore$ Possible integral values of λ are
$-7, -6, -5, -4, -3, 4, 5, 6, 7, 8$

$\therefore$ In all, there are 10 possible integral values of λ.

18 $[P + 1] = [P] + 1$. Let $[P] = n$, then n is integer.
$([P + 1], [P]) = (n + 1, n)$ lies inside the region of circles.

$S_1 = x^2 + y^2 - 2x - 15 = 0,$

$C_1 = (1, 0), r_1 = 4.$

and $\quad S_2 = x^2 + y^2 - 2x - 7 = 0,$

$C_2 = (1, 0), r_2 = 2\sqrt{2}$

Both circles are concentric.

$\therefore \quad (n + 1)^2 + n^2 - 2(n + 1) - 7 > 0$

and $(n + 1)^2 + n^2 - 2(n + 1) - 15 < 0$

$\Rightarrow \quad 4 < n^2 < 8$

which is not possible for any integer.

19 $L = 3x - 4y - k = 0$ touches the circle $S = x^2 + y^2 - 4x - 8y - 5 = 0$ at (a, b)

Tangent at (a, b) is

$ax + by - 2(x + a) - 4(y + b) - 5 = 0$

$T = x(a - 2) + y(b - 4) - 2a - 4b - 5 = 0$

Comparing $L = 0, T = 0$, we get

$$\frac{a - 2}{3} = \frac{b - 4}{-4} = \frac{2a + 4b + 5}{k}$$

Also, using $p = r$, we get

$\dfrac{6 - 10 - k}{5} = \pm 5 \Rightarrow k = 15$ or -35

Using $k = 15$, we get

$4a + 3b = 20$ and $9a - 12b = 45$

Solving $a = 5, b = 0$ and using

$k = -35$, we get $a = -1, b = 8$

We can verify that $(5, 0)$ and $(-1, 8)$ both satisfy equation of circle. Hence $k, (a, b) = 15\,(5, 0)$ or $-35, (-1, 8)$

20 Centre of the circle $x^2 + y^2 = a^2$ is (0, 0).

Let its reflection about the line $y = mx + c$ be (h, k).

Then, $(h/2, k/2)$ lies on this line.

$m\dfrac{h}{2} + c = k$ and $-\dfrac{k}{h} \times m = -1$

$\Rightarrow \quad mk = -h$

On solving these, we get

$h = \dfrac{2cm}{1 + m^2}, k = \dfrac{2c}{1 + m^2}$

Also, radius of reflected circle is a.

$\therefore$ Equation of relfected circle is

$$\left(x + \frac{2cm}{1 + m^2}\right)^2 + \left(y - \frac{2c}{1 + m^2}\right)^2 = a^2$$

$$\Rightarrow x^2 + y^2 + \frac{4\,cm}{1 + m^2}x - \frac{4\,c}{1 + m^2} + \frac{4c^2(1 + m^2)}{(1 + m^2)^2} - a^2 = 0$$

But given equation of reflected circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

On comparing, we get

$$\frac{g}{\frac{2\,cm}{1 + m^2}} = \frac{f}{\frac{-2c}{1 + m^2}} = \frac{c}{\frac{4c^2}{1 + m^2} - a^2} = 1$$

$\therefore \quad g = \dfrac{2cm}{1 + m^2}, f = -\dfrac{2c}{1 + m^2}, \dfrac{4c^2}{1 + m^2} - a^2 = c$